DISTRIBUTIVE JUSTICE QUANTIFIED

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In his famous paper, What is Justice, Kelsen has demonstrated clearly that the existing definitions and alleged principles of justice are empty or contradictory. However, quite recently some matehamaticians working in behavioral science, particularly in economics, have come forth with the claim that they can find "fair solutions" to problems in distributive justice, to the problems of "fair division." Thus they are trying what Spinoza suggested: to establish an *ethica ordine geometrico* and —without realizing it— to work our mathematical solutions to Aristotle's principle of "mesotes," the "middle way."

The problem of "fair division" means simply: how should a given amount of goods, especially a sum of dollars, (which can not be increased) be divided among two (or more) individuals (or groups of individuals) in a fair way. The term "fair" certainly means the same as "just" though some mathematical moralists try to make a distinction. The circle of individuals among whom the amount is to be divided, that is, those who have a "claim" to it, is assumed to be preditermined. We will restrict the discussion to two individuals.

That answers like *suum cuique* (to each his own) are too vague to lead to solutions is quite obvious. But the mathematical moralists want to derive precise solutions exclusively from the "utilities" of the parties which are to share in the available total. Now this is certainly not the only principle which can be applied. Another principle is to decide according to merits. One party may deserve to be rewarded on account of his past accomplishments. Another principle is to divide in the way which is considered best for the "society" in which the parties live although the society itself does not participate in the division.

Actually the laws governing cases of division vary considerably in the various legal orders. The laws determining how the inheritance of an individual who dies intestate should be distributed are considerably different in different legal orders; but, to my knowledge, there is no legal order which considers "utilities" in their regulation. If two people find a hidden treasure, some laws provide that it should be divided evenly among the finders, others that the owner of the ground where the treasure has been found should receive a part, or that a part has to be delivered to the state. Certainly, it can not be said that such rules are "unfair." However, there are some situations in which the circle of recipients is clearly defined and limited and where the principle of division should in some way depend on the utilities of the parties. Bargaining between employers and labor and the sharing of markets between duopolists are primary examples of such situations for which the mathematical moralists try to find solutions —assuming, of course, that "third interests," particularly the interests of the consumers are not to be taken into consideration.

If we thus restrict the problem to cases in which the circle of participants in the division is determined and in which the decision is to be found solely by considering their utilities, can a principle be found that yields specific solutions as to how the participants should share the total to be divided?

What are utilities? Obviously they are mental phenomena, satisfactions, mainly gratifications of needs, or, to put it in a simpler way, pleasures and the avoidance of pains, hence not accessible by direct observation. We are faced, therefore, with the old and much discussed problem of their measurement. I do not propose to go into this controversy. If it were possible to find cardinal measurements of utilities by applying a yardstick equally to different individuals both intrapersonal and interpersonal comparisons would be feasible. But we will see that in some situations measurements of a lower order are sufficient.

The Principle of Proportionally Equal Gratification (PEG)

I will propose now a principle by which some problems of division can be dealt with in a "satisfactory way." General consensus will be taken as the criterion of satisfaction with solutions. This criterion corresponds basically to the criterion of "truth" in observation in empirical science, wich rests on agreement on perceptions, on "protocol sentences" in the language of the logical positivists —such as: this fluid —here and now —is green, which will be accepted by everybody except those color blind. As we will see later, the mathematical moralists also use this criterion.

The principle states: A given amount of goods which is to be distributed justly among two people should be allocated to them in such a way that the ratio of gratificacion resulting from the share allocated should be the same for each party based on the gratification which the party would obtain if he received the available total.

This principle has already been stated by Aristotle. For him, virtue is a mean between two extremes of which one is excess and one is deficency. Thus, courage lies at the midpoint between cowardice and rashness. The moralist can find the virtue which he is looking for just as the geometer can find the point equally distant from either extreme, from either end of a line. (Don't we hear a mathematical moralist speaking?)

From this position, he argues concerning distributive justice that it "involves at least four terms, namely two persons for whom it is just and two shares which are just. There will be some equality between the shares as between the persons, since the ratio between the shares will be equal to the ratio between the persons. If the persons are not equal they will not have equal shares. The just is the proportionate."

We will introduce some simple symbols. We will designate the two parties "I" and "J," and their utility gains resulting from the division "u_i" and "u_j." The maximum gains if a party receives the total will be designated $\tilde{u}_i$ and $\tilde{u}_j$, their minima $\tilde{u}_i$ and $\tilde{u}_j$. These minimum gains will obviously be zero if no division takes place –neither party gets anything. PEG then is expressed as the equality of the two gratification ratios $G_i = G_j$.

$$\frac{\bar{u}_i}{\tilde{u}_i} = \frac{\bar{u}_j}{\tilde{u}_j} \qquad (1)$$

where "$\bar{u}_i$" and "$\bar{u}_j$" stand for the utility gains derived from the just amounts allocated to each party. This means that the relation between the "just" utility gains whould be the same as the relation between their maxima.

$$\frac{\bar{u}_i}{\bar{u}_j} = \frac{\tilde{u}_i}{\tilde{u}_j} \qquad (1a)$$

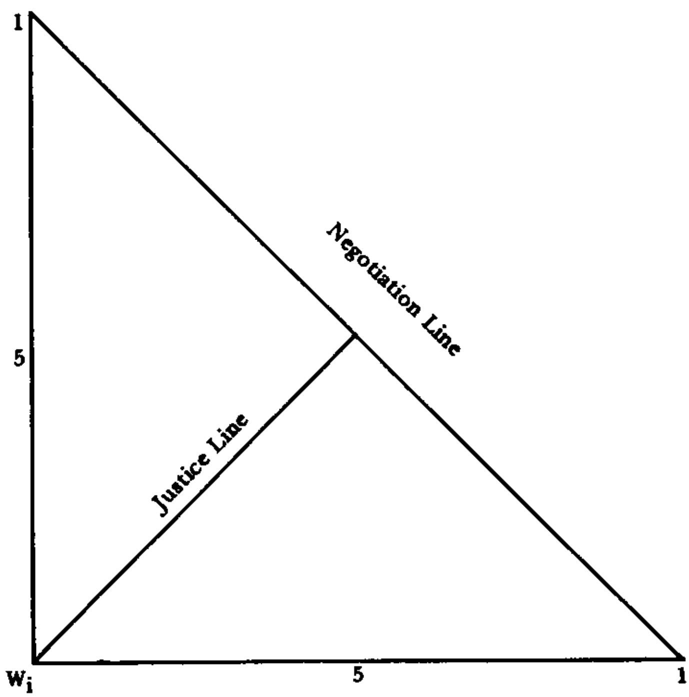
Linear Structure

We will now discuss the simplest case: "real" values and utilities are indentical; any unit of goods creates one unit of gratification for the party receiving it. The structure is linear; there is no "diminishing return" in utitility gains, any additional unit received adds the same utility gain, the utility gains from the first and last unit received are the same. The maximum gains —when either party receives the total available, therefore, are equal. We may "standardize" them by setting them at one.

We will now develop the method by which we arrive at the solution, the "just division" based on PEG. Though it seems obvious in this simple case, showing the method now will be helpful for its application in more complex cases.

We can present the situation in the graph below in which u_i is shown on the abscissa and u_j on the ordinate. The straight line connecting the two maxima represents all possible divisions; that is, all possible combinations of u_i and u_j. Obviously in this case the sum of each combination $u_i u_j$ adds to one since what one party gets more of the other gets less.*

Graph 1



* It is not proper to say that the utility gain for one party equals the utility loss for the other party. Both parties gain from division. It would be better to use the old legal term, *lucrum cessans* (lost gain) for the party which receives less when there is a shift in division.

The equation representing the (antidiagonal) line is:

$$u_j = 1 - u_i \tag{2}$$

We may call it the *"utility relation line"* or the *"negotiation line."***

Our problem is now to determine the point of just division, the "just point" on this line, so that $\frac{\bar{u}_i}{\tilde{u}_i} = \frac{\bar{u}_j}{\tilde{u}_j}$. Since in our case the maxima were set at one, the requirement is fulfilled by all points where $u_i = u_j$, that is, the line which ascends diagonally from the point of origin (0,0). The line may be called the *"justice line."* The "just point" lies at the intersection of the two lines where $u_i = u_j = .5$. The gratification ratios are one half for either party. The gratification ratios do not change, regardless of how large the total is, since we defined and set at one the maximum gratifications resulting from allocating the total to one party.

The division ratios D_i and D_j by which the share of each party is determined are one half. The available total is divided equally by the parties. To be sure, this is a "trivial" solution, but triviality in this sense certainly is identical with general consensus.

Suppose now real values are not identical with utilities for one party. Let us assume that for party I real values and utilities are equal, but that they are not so for party J. We still assume that the structure is linear, so that utilities are proportional to be real values (i and j); any real unit creates a multiple value in utility for J. $u_j = b_j$, $u_i = i$.

Suppose now that a total of 100 units is to be divided; we measure or estimate that J gets three times as much gratification from any actual unit as its real value. The parties decide to divide the total equally so that each receives 50 actual units. Thus J obtains 3 times 50 units of gratification; this is 150 which is one half of his maximum utility while I's maximum utility equals 100 so that he also receives one half of his maximum. The principle of PEG has been fulfilled.

In preparation for more complex cases we can show the application of PEG algebraically:

$$u_j = \tilde{u}_j - bu_i \quad \text{(negotiation line)}$$

$$u_j = \frac{\tilde{u}_j}{\tilde{u}_i}\, u_i \quad \text{(justice line)} \tag{3}$$

** This line is what is called in economics the "Pareto optimal" which simply means that any point in the graph below the line allocates party J less than the point vertically above on the line. The sum of u_i and u_j would then be less than 1, and the total would not be fully divided.

But $b = \frac{u_j}{u_i}$ since it expresses the slope of the line, the constant relation between any u_j and u_i, hence also for the maxima.

The solution, therefore, is

$$0 = \bar{u}_j - 2\frac{\bar{u}_j}{\bar{u}_i} u_i$$

$$\bar{u}_i = \frac{\bar{u}_i}{1}, \quad \bar{u}_j = \frac{\bar{u}_j}{2}$$

Now we must translate the utilities into the real values in order to carry out the actual division. This is easy as in our case real values and utilities are identical for I so that we simply have to subtract i from the total which leaves for J: 100 – 50 = 50 units. The same result can, of course, also be obtained from the formula u = b j, so that j = u/b, in our example 150/3 = 50.

To show another example: if the total to be divided is 300 units and b equals 4, party I receives 150 actual units and J, from receiving the remaining 150 units, obtains 600 units of gratification, which is one half of his maximum. (4 x 300 = 1200).

But can we expect general consensus for this solution? Party I —and many others with him— will argue: why should you, J, obtain a higher gratification than I? To which J, of course, may answer: what concern is it to you that I am so modest and easy to satisfy as long as we receive the same amount? Thus, party I will propose another principle of division: the utilities obtained from the division should be absolutely rather than proportionally equal: $\bar{u}_i = \bar{u}_j$. This solution favors party I, as he needs as larger amount of goods than J to obtain the same gratification.

Inserting $u_j = u_i$ into the negotiation equation we have

$$u_i = \bar{u}_j - bu_i$$

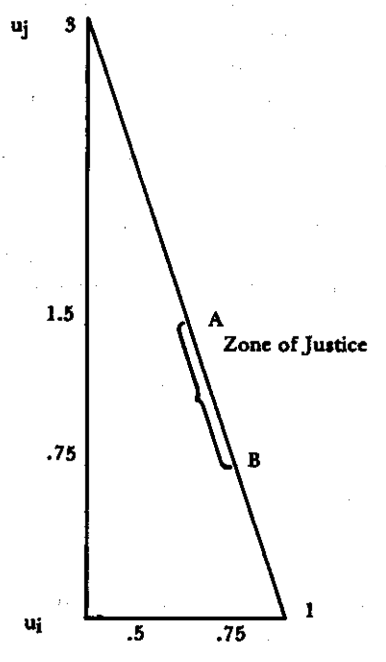
$$\bar{u}_i = \frac{\bar{u}_j}{1+b} \qquad (4)$$

Thus $\bar{u}_j = \frac{300}{1+3} = 75$ units.

To translate the utilities into real values is easy again when $i = u_i$ by subracting it from the total. Party I receives 75 units, J the remainder, 25 units.

The two solutions are shown in graph 2. Point A gives the solution for proportionality, point B for absolutely equal utilities. We have not found a unique solution but we have accomplished something: we have established a *"zone of justice"* between the two points. Any division to the left of A and any to the right of B would be rejected generally as unjust. A division by which J would receive both a higher gratification ratio and more of real value then I would certainly be rejected by any arbitrator, and so would a division by which J would receive less in utility gains than I. To give a full numerical example: suppose J's gratification is three times I's and 100 units are to be divide. At point A, $u_i = .5$, $u_j = 1.5$; $i = 50$, $j = 50$; $G_i = 1/2 = G_j$; $D_i = 1/2 = D_j$. At point B, $u_i = .75$, $u_j = .75$; $i = 75$, $j = 25$; $G_u = 3/4$, $G_j = 1/4$; $D_i = 3/4$, $D_j = 1/4$.

Graph 2



At A: $G_i = G_j$, $\bar{u}_i = \bar{u}_j$ $\quad i = j$, $D_i = D_j$,

At B: $G_i \neq G_j$, $\bar{u}_i = \bar{u}_j$ $\quad i \neq j$, $D_i \neq D_j$, To the left of A, party J would receive more than one half of the total and his gratification ratio would be higher than I's, to the right of B, J would receive even less than 25 units of gratification.

Thus PEG must be supplemented.* It serves only to determine the upper limit for J's share (and the lower limit for I's share), while the opposite limit of the zone of justice is derived from the principle that the utility gains resulting from the division should be equal. We may say that we can determine what is unjust, but not which solution within the zone of justice is *the* correct one.

A tentative solution to determine the just point within the zone of justice might be the midpoint between the ratios of 1/2 and 1/4 for J. Thus the *just gratification for J would* be 3/8, $u_i = 1.125$, and J would receive 37-1/2 units.

PEG and the Approach of the Mathematical Moralists

The mathematical moralists use what they call the *"abstract axiomatic method,"* that is, they suggest certain axioms (or postulates) and from them deduce the solution. I refer to their writings as to these axioms and the development of the solution. For our purpose it is sufficient to compare their solutions with PEG. The solution suggested by the mathematicians is: the fair division of a given amount ("an initial bundle of goods") lies at the point where the product of the utility gains resulting from the division is at its maximum.

But this is exactly the same point as the one derived from PEG. It is obvious when the maxima are (or are considered) equal; in this case the midpoint of the negotiation line connecting the two maxima is the point where the product of u_i and u_j is largest. The products drop from this point in both directions toward zero. But this is also true if the maxima are not equal. In our numerical example the product of u_i times u_j (that is $.5 \times 1.5 = .75$), is larger than the adjacent products; for example, if $u_i = .4$, $u_j = 3\text{-}3 \times .4 = 1.8$ and the product, $.4 \times 1.8 = .72$ is smaller than .75. Similarly when $u_i =$o .6. But this "just point" is the point where the gratification rations are equal.

What is the meaning of this theory in the view of its adherents? On this point the writers are not in agreement. Some think that the solution is "normative," others that the theory offers a "model" of rational behavior or its result, still others that it has predictive value.

* As long as $u_i = 1$: for the first solution $D_i = D_j = G_i = G_j = 1/2$ for the second solution $D_i \neq D_j$, $G_i \neq G_j$ but $D_i = G_i$, $D_j = G_j \neq 1/2$ (in linear structure).

Harsanyi has shown clearly how the solution (the maximum product of utility gains) can be derived from one —or perhaps the— leading principle of rationality, namely, that the rational decision between possible alternative courses of action should be made by comparing the products of the utility gains resulting from a specific course of action times their probabilities.

Thus, the remarkable thing is that the rational approach and the normative approach (PEG) lead to the same result, the maximization of the product of the utility gains, which means that in this respect virtue and knowledge coincide. Now if the task of a *mediator* is to find a rational solution, that is, the solution which has the best chance of being accepted if the parties act rationally, while the task of the *arbitrator* is to find a just solution (the "normative" aproach), the two solutions are the same (if the arbitrator wants to base his decision on PEG rather than on absolute equality).

Non-linear Structure

Let us turn non to a discussion of a situation in which J's utility gains resulting from additional units of real values decline. This is the usual pattern of diminishing returns — an additional unit adds a smaller amount of gratification than the previous one. For the rich man a dollar is a dollar, to the poor man the first dollar received means more than the second.

Assuming (or considering) the maximum gratifications equal and setting them at one, the situation is presented in graph 3. The negotiation line is curvilinear. The solution for equal utilities is found at the intersection between the negotiation line and the line where u_j equals u_j, that is, the ascending diagonal.

Here is a numerical example assuming parabolic structure:

I receives i	J receives j	Utility gains for I	Utility gains for J
(as ratios of total)			
D_i	D_j		
0	1	0	1
.25	.75	.25	.90
.50	.50	.50	.70
.75	.25	.75	.40
1	0	1	0

The solution is derived from the two equations:

$u_j = 1 - .2u_i - .8u^2{}_i$ and $u_i = u_j$

It lies at about. 60 (Point A in graph 3)

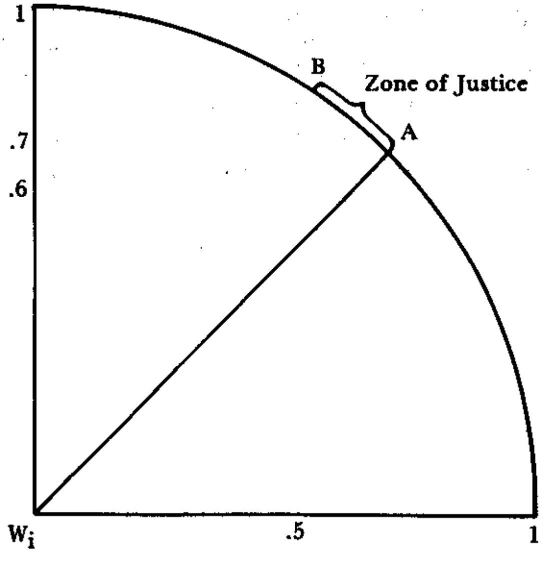
Thus, party I would receive about 60 units out of 100, obtaining a utility gain of .6, while J would be allocated 40 units only which, however, would give him the same gratification (ratio), namely .6, as I. This is also the solution suggested by the mathematical moralists, as at this point the product of the two utilities is at its maximum.

But now we are faced with the same problem as in the case of unequal maximum gratifications. Again J, the poor man, will object and argue: "why should the poor man be punished and recieve less than the rich man? It is irrelevant that the poor man obtains a higher gratification ratio than the rich from an equal amount." Thus the solution would not find general consensus, the poor man would feel unjustly treated and those sympathising with the underdog will side with him. He will suggest that the total should be divieded equally between the parties. This division gives J a higher gratification ratio (.7) than I (.5) as seen in the numerical example. (Point B in graph 3)

Thus again, we have established a zone of justice. A solution beyond the two points (A and B) would be rejected generally. The acceptable solution lies between the two points and again a compromise can be worked out within the zone.

A special case deserves a brief discussion: the case of the "broken negotiation line". Suppose I's utility gains coincide with the real values but J's utility line is linear up to some amount of goods smaller than the total available and then flattens out completely. This is the case when J does not need all the available goods. He is fully satisfied with less than the total. For example, there is a pile of perishable goods and J can not consume all of them (and, of course, can not use them in any other way by selling or giving them to others).

Graph 3



The graph shows a truncated triangle. If party I gets less than u_i at the point where the line changes direction, J gets more, but these additional goods do not increase his gratification.

The just point by applying PEG lies where $u_i = u_j$.

The solution in the case presented by graph 4 is:

The equation for the sloped line is:

$$u_j = 2.5 - 2.5u_i$$

When $u_j = u_i$

$$\bar{u}_i = 2.5. - 2.5\,\bar{u}_i$$

$$3.5\bar{u}_i = 2.5\ \bar{u}_i = .71\ \text{(approx.)} = \bar{u}_j$$

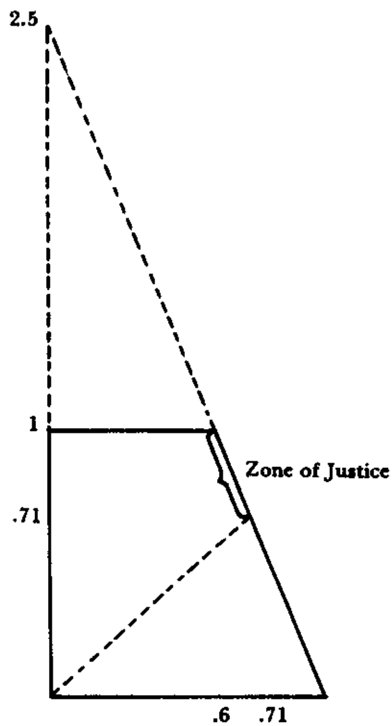
When we transform J's utilities into real values he receives less than I (as in the case of curviliniarity). I receives. 6 and J what is left, that is .4, which gives him the same gratification as I as

$$\tilde{u}_j = 2.5 - 2.5 \times .71 = .71$$

Any points right of this point A would not be acceptable since there could not be any reason why party I should obtain a higher gratificacion than J. However, while it might be argued that J should obtain a higher gratification than I, none would suggest that J should receive more than .4, since additional units would not increase his gratification and would be wasted, hurting I without helping J. At this point (B in the graph) the product of the utilities is at its maximum and this is the solution suggested by the mathematical moralist.

Thus, the zone of justice lies between the two points. A compromise within the zone may be worked out.

Graph 4



Unequal Minima and Maxima

Up to now we have discussed situations in which the initial situation of the two parties, the origin, was the same; we set the minima at zero. Now we will consider a situation in which these minima are not equal. We will restrict the discussion to the simplest case: real values and gratifications are equal, the structure is linear, and we set the total to be divided at one, and "standardize" the minima, too. We designate the increments in values (over the minima) v_i and v_j; u, therefore, $= \tilde{u}+v$.

There are now several possible solutions. In the first place we may simply ignore the initial difference, consider it irrelevant. We apply PEG to the increments; I as well as J receives one half of the total available $v_i = v_j = .5$. The gratification ratios based on the increments only are, of course, .5, too. However, the gratification ratios based on total utilities (initial plus gains) are unequal. Suppose

$$\left.\begin{aligned} \tilde{u}_i = .2 \text{ and } \tilde{u}_j = .1,\quad G_i &= \frac{.2+.5}{.2+\ 1} = \frac{.7}{1.2} = .58 \\ G_1 &= \frac{.1+.5}{.1+\ 1} = \frac{.6}{1.1} = .54 \end{aligned}\right\} \text{Point A in graph 5}$$

A second possible solution can be found by applying PEG to the total utilities rather than to their increments, (to u rather than v).

$$\frac{\bar{u}_i}{\tilde{u}_i} = \frac{\bar{u}_j}{\tilde{u}_j}$$

$$\frac{\tilde{u}_i + \bar{v}_i}{\tilde{u}_i} = \frac{\tilde{u}_j + 1 - \bar{v}_i}{\tilde{u}_j}$$

$$\frac{\tilde{u}_i + \bar{v}_i}{\tilde{u}_i} = \frac{\tilde{u}_j^* - \bar{v}_i}{\tilde{u}_j}$$

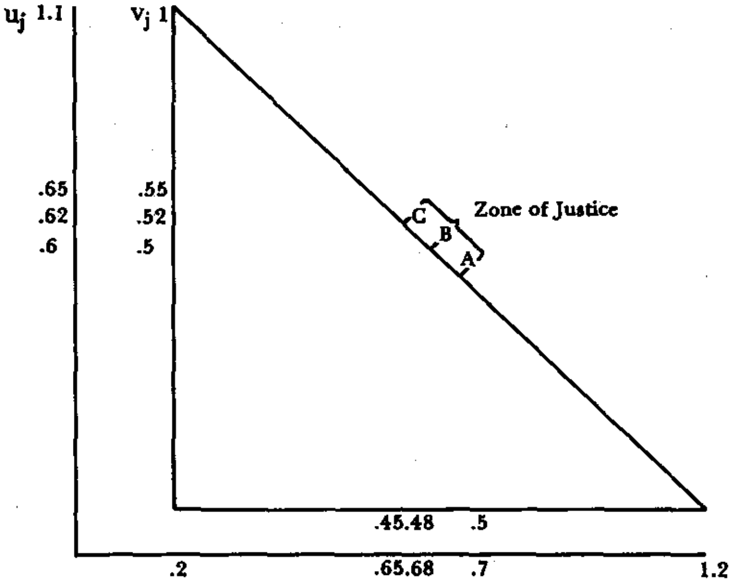
$$\bar{v}_i = \frac{\tilde{u}_j\,(\tilde{u}_i - \tilde{u}_i)}{u_j + u_i}$$

But $\tilde{u}_i - \tilde{u} = 1$

$$\bar{v}_i = \frac{\tilde{u}_j}{\tilde{u}_j + \tilde{u}_i} \qquad (5)$$

In our example $\bar{v}_i = .84$ and $\bar{v}_j = .52$ (appr.)

$$\bar{u}_i = .68 \text{ and } \bar{u}_j = .62$$



$$G_i = \frac{.68}{1.2} = .57 \text{ (appr.) and equals } G_j = \frac{.62}{1.1} \text{ (Point B)}$$

A third possibility is complete equalization of the utilities after the division; the party that is at a disadvantage at the start should receive more, so that $\bar{u}_i = \bar{u}_j$

$$\bar{u}_i + \bar{v}_i = \bar{u}_j + \bar{v}_j$$

But $v_j = 1 - v_i$

$$\bar{u}_i + \bar{v}_i = \bar{u}_j + 1 - \bar{v}_i$$

$$2\bar{v}_i = 1 + u_j - \bar{u}_i$$

$$\bar{v}_i = \frac{1 + u_j - u_i}{2}$$

Or: since $1 + u_j = u_j$

$$\bar{v}_i = \frac{u_j - u_i}{2}$$

$$\bar{v}_i = \frac{1 + .1 - .2}{2} = .45,\ \bar{v}_j = .55,\ \bar{u}_i = .45 + .2 = .65 = \bar{u}_j = .55 + .1$$

(Point C)

In other words, each party receives half of the total available plus or minus half of the difference in the initial positions. The utility gratification ratios are different

$$G_i = \frac{.65}{1.2} = .54 \qquad G_j = \frac{.65}{1.1} = .59$$

Summarizing,

	Point A PEG for increments	Point B PEG for utilities	Point C Equal (final utilities
v_i	.5	.48	.45
v_j	.5	.52	.55
u_i	.7	.68	.65
u_j	.6	.62	.65
G_i	.58	.57	.54
G_j	.54	57	.58

We have not found a unique solution, but we have again established a zone of justice. It lies between the two extreme solutions (Points A and C in the graph); that is, between the solution in which the initial difference is maintained and in which it is completely eliminated. Solutions to the right of A and to the left of C could hardly be asserted meaningfully as just solutions. A solution to the right of A would mean that the gap between the final utilities would be even larger than at the start. A solution to the left of C would mean that I, who had an initial advantage, would end up with less than J.

Tenative compromises within the zone may be suggested. Position B would serve as such a compromise.

It is not only the minima which create a problem — a parallel problem appears when we consider the maxima. We have defined *maximum gratifications* as those which result from allocation of the total to the specific party. But the gratification obtained from receiving the total available does not always mean complete gratification, saturation of the need (s) which the goods to be distributed can fulfill. Should the extent to which these needs are satisfied be considered in a theory of division of utilities? After all, utility theory originated from considerations of needs, from Gossen's scale of needs. But the extent to which the parties needs are satisfied from the allocation of the total may be different. A big man may not be able to still his hunger when consuming a given amount of food, while a small man may be completely saturated — even from consumption of a part of it.

Thus we may set complete gratification rather than gratification from receiving the total as the basis for computing the gratification ratios. This would mean, following the first part of Marx' postulate: to each according to his needs (to each according to his abilities).

A division applying PEG using these ratios would be different from the division using gratification ratios based on gratification from receiving the total. A man who needs twice as much food as another would have to receive a larger share of the food available, even if the gratifications obtained from receiving the total available were equal. It would be possible to work out a compromise between the two different gratification ratios. But would (and should) an arbitrator be guided by such considerations? Should he let considerations of the great needs, the high goals of the one party, guide him uncritically in his decision? This would mean that the greedy, ambitious individual would be favored, while the modest individual would be punished. The miser whose complete gratification would be obtained only from receiving an inordinately high amount —possibly infinite— of money would be favored against the individual who has a "realistic" aim. Thus a new element enters into the theory. Utility theory applied to "fair division" can deal only with "legitimate" interests and utilities. How far "legitimacy" extends can not be decided even if cardinal measurements of utilities are possible. The mediator may disregard the problem but the arbitrator can not.